ON THE EXISTENCE OF SHOCK WAVES OF SMALL INTENSITY

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As is well known from the theory of shock waves in ordinary and magnetohydrodynamics, the discontinuities of hydrodynamic quantities at the transition through the front of a wave must satisfy certain well-defined algebraic relationships. Furthermore, it is also known that not every discontinuity satisfying these relations actually represents a shock wave. When dealing with applications it is important to know which discontinuities represent a shock wave and which do not.

Let us assume that to a given particular discontinuity of the hydrodynamic quantities $u_- \to u_+$ there corresponds a steady shock wave moving with velocity U. This means that the system of equations of magnetohydrodynamics, which takes into account the dissipative processes, must have solutions of the form $u = u(x - Ut)$, which tend to u_- and u_+, when $x \to \pm\infty$. Let us designate these solutions as transitional.

The problem concerning the sufficiency of conditions of existence of the transitional solution in the system of nonlinear equations was formulated by Gel'fand [1]. Germain [2] has proven the existence of a fast shock wave in magnetohydrodynamics in the case of plane flow. Under more restrictive assumptions he also established the existence of the slow shock wave. Kulikovskii and Liubimov [3] and Liubimov [4] have proven the existence of fast and slow shock waves under the assumption that only a part of the coefficients of dissipation are different from zero. A new approach to the problem of the existence of the transitional solutions was proposed by Godunov [5,6]. The present paper treats dissipative systems of nonlinear equations. Particular cases of such systems are the equations of ordinary and magnetohydrodynamics for which the dissipative processes are taken into account.

1. Statement of the problem. We shall consider discontinuities of sufficiently small magnitude. The velocity of transition of such a discontinuity closely equals the phase velocities V of the system. It is natural to distinguish between ordinary and singular phase velocities. For example, in magnetohydrodynamics ordinary velocities are the velocities of slow and fast magneto-sound waves, whereas the Alfven velocity is a singular velocity. A precise definition of a dissipative system and of an ordinary phase velocity will be given below.

In the case of dissipative systems the following statement will be proven:

To any discontinuous solution of a system, formulated without taking into account the dissipative processes, there corresponds a shock wave, i.e. a transitional solution of an exact system, if (a) the discontinuity of a discontinuous solution is stable with regard to attenuation, (b) the velocity U of transition of the discontinuity is sufficiently close to one of the ordinary phase velocities and (c) the discontinuity is sufficiently small.

If the foregoing conditions are fulfilled, the existence of slow and fast shock waves of sufficiently small intensity in magnetohydrodynamics is established immediately, regardless of the value of the coefficients of dissipation.

The proof is based on the following theorem.

Theorem 1. The system

$$a_m y^{(m)} + \ldots + a_1 y' + \varepsilon\beta_0 (y - y^2) = \varepsilon \sum_{k=1}^{m-1} y^{(k)} \varphi_k (y, y', \ldots, y^{(m-1)}, \xi; \varepsilon) + \varepsilon^2 F_1 (y, y', \ldots, y^{(m-1)}, \xi; \varepsilon) \qquad (1.1)$$

$$\xi_i' - \sum_{k=2}^{m} b_{ik}\xi_k = f_i(y) + \varepsilon F_i (y, \xi; \varepsilon) \qquad (i=2, 3, \ldots, m)$$

$$\xi_i = f_i(y) + \varepsilon F_i (y, \xi; \varepsilon) \qquad (i = m+1, \ldots, n)$$

has a transitional solution $y(x, \varepsilon)$, $\xi(x, \varepsilon) = \{\xi_i(x, \varepsilon)\}_2^n$, which satisfies the conditions $y(-\infty, \varepsilon) = 0$, $y(+\infty, \varepsilon) = 1 + O(\varepsilon^2)$, if all the roots of the polynomial $a_m\nu^m + a_{m-1}\nu^{m-1} + \ldots + a_1\nu$ are real and simple, if the determinant $|b_{ik}|_2^m$ is not equal to zero, if all the functions F_i $(i = 1, \ldots, n)$, f_i $(i = 2, 3, \ldots, n)$ and φ_k $(k = 1, \ldots, m - 1)$ are analytic with respect to their arguments and if the absolute value of the parameter ε is sufficiently small.

The proof of this theorem is given in the appendix.

2. Dissipative and ideal systems. The systems of equations of ordinary and magnetohydrodynamics may be written schematically in one-dimensional form as follows:

$$\frac{\partial}{\partial t} a_i(u) + \frac{\partial}{\partial x} b_i(u) = 0 \qquad (i = 1, \ldots, k_1)$$

$$\frac{\partial}{\partial t} a_i(u) + \frac{\partial}{\partial x} b_i(u) = \sigma_i \psi_i(u) \equiv c_i(u) \qquad (i = k_1 + 1, \ldots, k_2) \tag{2.1}$$

$$\frac{\partial}{\partial x} b_i(u) = \sigma_i \psi_i(u) \equiv c_i(u) \qquad (i = k_2 + 1, \ldots, n)$$

where $u = \{u_j\}_1^n$ is a set of quantities which describe the condition of the fluid (pressure, temperature, velocity components, magnetic field and others), $a_i(u)$, $b_i(u)$ and $c_i(u)$ are certain analytical functions, $\sigma_i^{-1} > 0$ are coefficients of dissipation, namely viscosity, magnetic viscosity, thermal conductivity.

The system (2.1) admits of solutions of the form $u = u^\circ$ which describe constant homogeneous flows. Any vector may be substituted for u°, provided it satisfies the system of equations $\psi_i(u^\circ) = 0 (i = k_1 + 1, \ldots, n)$.

Let the system (2.1) be called a dissipating system if all of its solutions are asymptotically stable with respect to small disturbances of the initial condition.

If system (2.1) is dissipative, then it is easily seen that all the roots $\omega_s(k)$ $(-\infty < k < \infty)$ of equation

$$D(i\omega, ik, u^\circ) \equiv \det | i\omega a_{ij}(u^\circ) - ikb_{ij}(u^\circ) - \sigma_i \psi_{ij}(u^\circ) | = 0$$

$$a_{ij}(u^\circ) = \left.\frac{\partial a_i}{\partial u_j}\right|_{u=u^\circ} \text{and so on.} \tag{2.2}$$

are located in the upper half-plane.

In ordinary and magnetohydrodynamic fluids all free vibrations vanish with time. Therefore the systems of differential equations by which they are defined are dissipative.

If in one of the equations (2.1) we assume $\sigma_i = \infty$ or $\sigma_i = 0$, then that equation assumes the following form

$$\psi_i(u) = 0, \quad \frac{\partial}{\partial t} a_i(u) + \frac{\partial}{\partial x} b_i(u) = 0 \quad \text{or} \quad \frac{\partial}{\partial x} b_i(u) = 0$$

Assuming that a part of the coefficients σ_i vanish, while the rest tend to infinity, we will have an ideal system. Consequently in this manner there are 2^{n-k} separate ideal systems associated with the system (2.1). The ideal system in which all $\sigma_j = \infty$ $(\sigma_j = 0)$ we shall call the lowest (the highest) system.

We shall call two systems neighboring systems if they differ by only one equation.

If we assume that a part of the coefficients σ_j vanishes and that the rest equal infinity, then a part of the lines in the determinant (2.2) will be substituted by the lines $\psi_{ij}(u^o)$ and a part by the lines $i\omega a_{ij}(u^o) - ikb_{ij}(u^o)$. In this case the solutions of Equation (2.2) will have the following form: $\omega_s = V_s k$, where $V_s - V_s(u^o)$ are phase velocities of the corresponding ideal system. The phase velocities of the lowest system are roots of the equation

$$\Delta(V) \equiv \det \begin{vmatrix} -Va_{ij}(u^\circ) + b_{ij}(u^\circ) \\ \psi_{ij}(u^\circ) \end{vmatrix} = 0 \tag{2.3}$$

When referring to phase velocities of system (2.1) we mean the phase velocities of the lowest system.

Phase velocities of all ideal systems associated with the dissipative system (2.1) are real [8] (see also [9]).

3. Ordinary phase velocities. Let $V = V(u_-)$ be one of the phase velocities of system (2.1). We shall call it an ordinary velocity if the following conditions are fulfilled.

(a) The quantity V is a simple root of the characteristic equation (2.3) for $u^o = u_-$.

(b) In one of the neighboring systems all phase velocities differ* from V for $u^o = u_-$.

(c) The uppermost system has no phase velocity identical with V for $u^o = u_-$.

(d) To every value of U from any neighborhood of the number V there corresponds a discontinuity, propagating with velocity U and tending to zero as $U \to V$, while the ratio $(u_+ - u_-)/(U - V)$ tends to a finite limit for $(U - V) \to 0$.

(e) All roots ν of equation $D(-\nu V, -\nu, u_-) = 0$ are simple and real, with the exception of $(k_1 + 1)$ which is a multiple root $\nu = 0$.

In magnetohydrodynamics the Alfven and entropic waves [10] are singular waves because for them condition (d) is not fulfilled [11]. On the other hand, fast and slow magneto-sound waves are ordinary waves. Indeed, the uppermost system in this case has an infinite viscosity and thermal

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* If system (2.1) is dissipative then the property (a) is the result of property (b).

conductivity and zero electric conductivity. In such a system we have

$$\frac{\partial^2 H}{\partial x^2}=0, \qquad \frac{\partial^2 v}{\partial x^2}=0, \qquad \frac{\partial T}{\partial x}=0, \qquad \frac{\partial}{\partial t}\rho+v_x\frac{\partial}{\partial x}\rho=0$$

This means that one phase velocity equals v_x, and the rest are infinite. Condition (c) is fulfilled. To check condition (b) consider the neighboring system, in which the coefficient of thermal conductivity is infinite, the magnetic and ordinary viscosity vanish. Small oscillations of such a system are isothermal and the phase velocities may be obtained from the phase velocities of ideal magnetohydrodynamics by replacing the ordinary sound velocity by the isothermal sound velocity.

The most complicated task would be the checking of condition (e). However, the property of (e) is proved already in the work of Germain [2]. In the case of a sound wave in ordinary hydrodynamics condition (e) has been proven by Godunov [5].

4. The system which determines the structure of a shock wave. To the stationary shock wave, propagating with velocity U, there corresponds a solution of system (2.1) of the form $u(x, t) = u(x - Ut)$, where $u(\pm\infty) = u_\pm$, $u'(\pm\infty) = 0$. Evidently function $u(x)$ is the solution of the system of equations

$$\frac{d}{dx}\{-Ua(u)+b(u)\}=c(u) \qquad (4.1)$$

For the purpose of further considerations it is necessary to make certain transformations of system (4.1). Let us introduce the notations $B(u) = b(u) - V(u_-)a(u)$, and write Equation (4.1) in the following form:

$$\frac{d}{dx}\{-\varepsilon a(u)+B(u)\}=c(u) \qquad (\varepsilon=U-V(u_-)) \qquad (4.2)$$

The set of all vectors, in which the first k_1 (last $n - k_1$) of the coordinates vanish, will be denoted by $H_1(G_1)$. Let P_1 and Q_1 be the operators of the orthogonal projection onto the subspaces H_1 and G_1 respectively. These operators are given by the matrices

$$a_{ik}(u_-), \qquad B_{ik}(u_-)=\left.\frac{\partial B_i}{\partial u_k}\right|_{u=u_-}, \qquad c_{ik}(u_-)=\left.\frac{\partial c_i}{\partial u_k}\right|_{u=u_-}$$

which we shall denote by a_1, B_1 and c_1. Note the relations

$$Q_1c_1=Q_1c_1=0, \qquad P_1c=c, \qquad P_1c_1=c_1$$

From (4.2) it follows that $Q_1\{-\varepsilon a(u) + B(u)\} = \text{const}$. Without limiting the generality of the investigation, we shall assume $a(u_-) = 0$,

$B(u_-) = 0$. Also

$$-\varepsilon Q_1 a(u) + Q_1 B(u) = 0 \tag{4.3}$$

and Equation (4.2) then assumes the following form:

$$\frac{d}{dx}\{-\varepsilon P_1 a(u) + P_1 B(u)\} = R(u) - \varepsilon Q_1 a(u) \quad (R = Q_1 B + c) \tag{4.4}$$

Let us prove that for certain known conditions system (4.4) has a transitional solution. From (4.4) it follows that quantities $u_- = u(-\infty)$ and $u_+ = u(+\infty)$ satisfy relation

$$R(u_-) - \varepsilon Q_1 a(u_-) = R(u_+) - \varepsilon Q_1 a(u_+) = 0 \tag{4.5}$$

5. Discontinuities propagating with velocity close to the ordinary phase velocity. Let us investigate Equation (4.5) in detail. Let us assume $u_+ = u_- + \varepsilon v_+$ and expand the vector-functions $R(u_+)$ and $a(u_+)$ in Taylor series. We obtain

$$R_1 v_+ + \varepsilon R_2(v_+) + \ldots = \varepsilon Q_1 a_1 v_+ + \varepsilon^2 Q_1 a_1(v_+) + \ldots \tag{5.1}$$

Here R_1 is a particular or intrinsic operator. This follows directly from (2.3). Therefore there exists a vector v°, such that $R_1 v^\circ = 0$. Let us introduce in the investigation also vector v^*, which satisfies relation $R_1{}^* v^* = 0$, where $R_1{}^*$ is the operator conjugate to R_1. Notice, that

$$c_1 v^\circ = P_1 c_1 v^\circ = P_1 R_1 v^\circ = 0$$

According to property (d) the ordinary phase velocity vector v_+ tends to a finite limit, when $\varepsilon \to 0$. From (5.1) it follows that

$$\lim_{\varepsilon\to 0} v_+ = \mu v^\circ, \quad \text{or} \quad u_+ = u_- + \varepsilon\mu v^\circ + O(\varepsilon^2) \tag{5.2}$$

The number μ is easily found by multiplying Equation (5.1) by v^* and then approaching the limit $\varepsilon \to 0$; we obtain the result

$$\mu\,(v^*, R_2(v^\circ)) = (v^*, Q_1 a_1 v^\circ)$$

Let us prove that $(v^*, Q_1 a_1 v^\circ) \neq 0$. According to property (a) of an ordinary phase velocity the number $V_- = V(u_-)$ is a simple zero of function $\Delta(V) = \det |R_1 - (V - V_-) Q_1 a_1|$. Therefore $\Delta'(V_-) \neq 0$. On the other hand

$$\Delta'(V_-) = -\sum_{i,k=1}^{n} (Q_1 a_1)_{ik} \Gamma_{ik} \tag{5.3}$$

where Γ_{ik} are algebraic supplements of determinant det $|R_1|$.

Since according to (5.3) not all Γ_{ik} vanish and since the operator R_1 is particular, the numbers Γ_{ik} may be represented in the form $\Gamma_{ik} = v_{i*}v_k^\circ$, if the vectors v° and v^* are suitably normalized. Substituting this expression of Γ_{ik} in (5.3) we obtain

$$\Delta'(V_-) = -(v^*, Q_1 a_1 v^\circ) \neq 0$$

Consequently

$$(v^*, R_2(v^\circ)) \neq 0. \qquad \mu = (v^*, Q_1 a_1 v^\circ) / (v^*, R_2(v^\circ))$$

6. Reduction of system (4.3) to (4.4) to the canonical form. Let us note that $dB(u)/dx = \dot{B}(u)du/dx$. According to property (c) of an ordinary phase velocity the operator $\dot{B}(u_-) - B_1$ has a reverse operator. Therefore for sufficiently small $|\varepsilon|$ and $|u - u_-|$ Equation (4.2) may be rewritten in the form

$$\frac{du}{dx} = [\dot{B}(u) - \varepsilon\dot{a}(u)]^{-1} c(u) \tag{6.1}$$

Introducing the notations $H_2 = B_1^{-1}H_1$ and $G_2 = B_1^{-1}G_1$, we shall define the operator P_2, assuming

$$P_2\varphi = \varphi \quad (\varphi \in H_2), \qquad P_2\varphi = 0 \quad (\varphi \in G_2)$$

To simplify further derivations we shall assume, without limiting the generality, $u_- = 0$. Let us separate the desired function $u(x)$ into two terms

$$u = z + w \qquad (z \in H_2,\ w \in G_2)$$

and substitute it in (4.3)

$$Q_1 B_1 z + Q_1 B_1 w = \varepsilon Q_1 a(u) + Q_1 [B_1 u - B(u)]$$

Since $z \in H_2$, then $B_1 z \in H_1$ and $Q_1 B_1 z = 0$. Further, $w \in G_2$, $B_1 w \in G_1$ and $Q_1 B_1 w = B_1 w$. Therefore

$$w = B_1^{-1} Q_1 \{\varepsilon a(u) + B_1 u - B(u)\} \tag{6.2}$$

Apply the operator P_2 to both parts of (6.1).

$$\frac{dz}{dx} = P_2 B_1^{-1} c_1 u + P_2 \{[\dot{B}(u) - \varepsilon\dot{a}(u)]^{-1} c(u) - B_1^{-1} c_1 u\}$$

Note that $c_1 u \in H_1$, $B_1^{-1} c_1 u \in H_2$. Therefore $P_2 B_1^{-1} c_1 u = B_1^{-1} c_1 u$ and

$$\frac{dz}{dx} = B_1^{-1} c_1 u + F(u), \qquad F(u) = P_2 \{[\dot{B}(u) - \varepsilon \dot{a}(u)]^{-1} c(u) - B_1^{-1} c_1 u\} \quad (6.3)$$

System (6.2) to (6.3) is equivalent to the initial system (4.1).

Before proceeding further let us prove that $v^0 \in H_2$. Starting from equation $R_1 v^0 = 0$, we obtain

$$0 = Q_1 R_1 v^\circ = Q_1 (Q_1 B + c_1) v^\circ = Q_1 B v^\circ \qquad (6.4)$$

Hence if follows, that $B_1 v^0 \in H_1$ and $v^0 \in B_1^{-1} H_1 = H_2$.

Let $e_1, e_2, \ldots, e_m$ $(m = n - k_1)$ be any starting base in H_2, where $e_1 = v^0$. In addition, let there be a certain starting base $e_{m+1}, \ldots, e_n$ in G_2. Assume

$$z = \varepsilon \mu y e_1 + \sum_{i=2}^{m} \varepsilon^2 \xi_i e_i, \qquad w = \sum_{i=m+1}^{n} \varepsilon^2 \xi_i e_i \qquad (6.5)$$

If in Equations (6.2), (6.3) coordinates are introduced, a system of the following form is obtained

$$y' = \varepsilon \sum_{j=2}^{m} \beta_{1j} \xi_j + \varepsilon (\alpha y + b y)^2 + \varepsilon^2 \varphi (y, \xi_2, \ldots, \xi_n; \varepsilon) \qquad (6.6)$$

$$\xi_i' - \sum_{j=2}^{m} \beta_{ij} \xi_j = \alpha_i y + b_i y^2 + \varepsilon F_i (y, \xi_2, \ldots, \xi_n; \varepsilon) \qquad (i = 2, 3, \ldots, m)$$

$$\xi_i = \alpha_i y + b_i y^2 + \varepsilon F_i (y, \xi_2, \ldots, \xi_n; \varepsilon) \qquad (i = m+1, \ldots, n)$$

Let us represent by

$$D_1 (\nu, \varepsilon, u_-) = \det | \nu (-\varepsilon P_1 a_1 + P_1 B_1) - R_1 + \varepsilon Q_1 a_1 |$$

the characteristic polynomial of system (4.4) at the point $u = u_-$. System (6.6) is equivalent to the system (4.4). Therefore its characteristic polynomial is identical to $D_1(\nu, \varepsilon, u_-)$ except for the multiplier. In part we obtain

$$D_1 (\nu, 0, u_-) = \text{const}\, \nu \det | \beta_{ij} - \nu \delta_{ij} |_2^m \qquad (6.7)$$

Assume

$$D_1(\nu, 0, u_-) = a_m\nu^m + a_{m-1}\nu^{m-1} + \ldots + a_1\nu \equiv \nu\Delta_1(\nu)$$

Let us apply now the operator $\Delta(d/dx)$ to the first of equations (6.6). By the well known Hamilton-Kelly theorem [12], we obtain

$$D_1\left(\frac{d}{dx}, 0, u_-\right)y + \varepsilon \sum_{k=0}^{m-1} \beta_k y^{(k)} = \varepsilon \sum_{i,k=0}^{m-1} \gamma_{ik} y^{(i)} y^{(k)} +$$

$$+ \varepsilon^2 F_1(y, y', \ldots, y^{(m-1)}, \xi_2, \ldots, \xi_n; \varepsilon) \tag{6.8}$$

Let us denote by (A) the system which will be obtained, if in the system (6.6) the first equation is replaced by Equation (6.8). Systems (6.6) and (A) are equivalent in the sense that every transitional solution of one is the transitional solution of the other. This follows from the fact that among the solutions of equation $\Delta_1(d/dx)\varphi = 0$ only the trivial solution $\varphi \equiv 0$ is bounded on the entire x-axis.

7. Existence of shock waves. It will be shown that system (A) is a particular case of system (1.1) and that it satisfies all the conditions of Theorem 1.1.

Condition (a) of the Theorem 1.1 is satisfied for sufficiently small values of absolute magnitude of ε because of the condition (e) pertaining to the ordinary phase velocity, since

$$D_1(\nu, 0, u_-) = (-\nu) \quad D(-\nu V_-, -\nu, u_-)$$

To check condition (b), use is made of the following property of the dissipating systems. Let $\nu = \nu_0(U)$ be the root of equation $(-\nu)^{-k_1} D(-\nu U, -\nu, u_-) = 0$, which becomes zero for $U = V_-$.

If velocity V_- possesses the property (b) of the ordinary phase velocity*, then $\nu_0'(V_-) < 0$.

It is easily seen that the characteristic polynomial of the system (A) has the root $\nu(\varepsilon) = -\varepsilon\beta_0/a_1 + O(\varepsilon^2)$ for small ε. Therefore, $\beta_0/a_1 > 0$.

According to the condition of stability of a discontinuity relative

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* This statement is proven in [9] for systems of a more special form than system (2.1). However, the derived proof is valid in the case of the systems of the form (2.1).

to dissipation, the following inequality must be satisfied

$$V(u_+) < U < V(u_-)$$

i.e. it must be that $\varepsilon < 0$. And thus $\varepsilon\beta_0 a_1 < 0$ and the condition (b) is satisfied. At the same time it was proved that condition (c) is also satisfied, since on account of (6.7)

$$a_1 = \mathrm{const}\det|\beta_{ik}|_2^m$$

It remains to prove that $\gamma_{00} = \beta_0$. To do this we notice that the coordinates of vector u_+ according to (5.2) equal

$$y_+ = 1 + O(\varepsilon), \qquad \xi_i = O(\varepsilon) \qquad (i = 2, 3, \ldots, n)$$

On the other hand, a constant vector $u = u_+$ is the solution of the system (A). Substituting it into Equation (6.8), we obtain

$$\varepsilon\beta_0 = \varepsilon\gamma_{00} + O(\varepsilon^2)$$

Hence, it is concluded that $\gamma_{00} = \beta_0$.

Thus, the statement formulated in Section 1 has been proven.

In particular it has been proven that there exists a structure in slow and fast magnetohydrodynamic waves of sufficiently small intensity.

8. ***Appendix.*** **Auxiliary equation.** We shall begin the proof of Theorem (1.1) with the investigation of the auxiliary equation

$$P\left(\frac{d}{dx}\right)y + \varepsilon f(y) + \varepsilon\psi(x) = 0 \qquad (0 < \varepsilon \leqslant 1) \tag{8.1}$$

Define two numbers $a_- > 0$ and $a_+ < 0$ in such manner that the polynomials $P(\nu) + a_\pm$ have only real and also simple zeros. In regard to functions $f(y)$ and $\psi(x)$ we assume the following.

(1) Function $f(y)$ is defined, continuous and differentiable in a closed interval $p_1 \leqslant y \leqslant p_2$, where $p_1 a_- = p_2 a_+$.

(2) Let

$$0 \leqslant f'(y) < a_- \quad (p_1 \leqslant y \leqslant 0) \qquad 0 \geqslant f'(y) > a_+ \quad (0 \leqslant y \leqslant p_2)$$

(3) No matter what the interval is (α, β) $(p_1 < \alpha < 0 < \beta < p_2)$, there exists a number $d(\alpha, \beta) > 0$, such that

$$|f'(y)| > d(\alpha,\ \beta)\,|a(y)| \quad (y \bar{\in} (\alpha,\ \beta)), \qquad a(y) = \begin{cases} a_-\ (y<0) \\ a_+\ (y>0) \end{cases}$$

(4) Function $\psi(x)$ is determined on the whole axis $-\infty < x < \infty$ and tends to certain limits $\psi(-\infty)$, $\psi(+\infty)$, when $x \to \pm\infty$.

(5) The following inequalities are valid

$$-\frac{1}{2f}(0) \leqslant \psi(x) \leqslant -f(p_1) \qquad (-\infty < x \leqslant 0)$$

$$-\frac{1}{2f}(0) \leqslant \psi(x) \leqslant -f(p_2) \qquad (0 \leqslant x < \infty)$$

Let us denote by S the set of all continuous functions $\omega(x)$ in the intervals $(-\infty,\ 0)$ and $(0,\ \infty)$, which satisfy the following inequalities

$$p_1 \leqslant \omega(x) \leqslant 0 \qquad (x<0), \qquad\qquad 0 \leqslant \omega(x) \leqslant p_2 \qquad (x>0)$$

The following theorem is valid:

Theorem 8.1. If conditions (1) to (5) are satisfied, then Equation (8.1) has in S a transitional solution and this solution is unique [7].

We shall denote by Ψ_f the set of all functions $\psi(x)$, which satisfy conditions (4) and (5). Theorem (8.1) shows that there exists an operator T_ε which transforms functions $\psi \in \Psi_f$ into the solutions $y(x) \in S$ of Equation (8.1)

$$y = T_\varepsilon \psi$$

The following theorem may be proven:

Theorem 8.2. There exists a number $\varepsilon_1 > 0$, such that for all ε in the interval $0 < \varepsilon < \varepsilon_1$, and for all ψ_1, $\psi_2 \in \Psi_f$ the following evaluation is valid

$$\| a(x)\,[T_\varepsilon\psi_2 - T_\varepsilon\psi_1] \| < \frac{4}{d(f)} \| \psi_2 - \psi_1 \|$$

$$d(f) \equiv d\left(-\frac{f(0)}{4a_-},\ \frac{f(0)}{4a_+}\right), \qquad \| \psi \| = \sup_x |\psi(x)|$$

The following consequence follows from Theorem 8.2.

Consequence 8.1. If all the conditions of Theorem 8.2 are satisfied and $0 < \varepsilon < \varepsilon_1$, then

$$\| y_2^{(k)} - y_1^{(k)} \| < \varepsilon M_k \left(1 + \frac{4}{d(f)}\right) \| \psi_2 - \psi_1 \| \qquad (k = 1, 2, \ldots, m-1)$$

where

$$\psi_i \in \Psi_f, \quad y_i = T_\varepsilon \psi_i \quad (i = 1, 2), \qquad M_k = \int_{-\infty}^{\infty} | Y_0^{(k-1)}(x) | \, dx, \; Y_0(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{ze^{zx}}{P(z)} dz$$

Proof of Theorem 1.1. According to the condition of the theorem the determinant $|b_{ik}|_2^m$ does not vanish. Therefore the system of equations

$$\xi_i' - \sum_{k=2}^{m} b_{ik} \xi_k = \psi_i(x) \qquad (\| \psi_i \| < K, \; i = 2, 3, \ldots, m)$$

has a bounded solution $\xi_i(x)$ along the entire axis. There exists a constant B such that

$$\| \xi_i \| < BK \qquad (i = 2, 3, \ldots, m)$$

Assume

$$\max | f_i(y) | = \mu_i \quad (-1 \leqslant y \leqslant 1) \qquad (i = 2, 3, \ldots, n)$$

and denote by D the region in $(n + m - 1)$-dimensional space of variables $y, y', y'', \ldots, y^{(m-1)}$; $\xi_2, \xi_3, \ldots, \xi_n$, determined by the inequalities

$$-1 < y < 1, \quad | y^{(k)} | < \varepsilon M_k, \; | \xi_i | < (B + 1) \mu_i \quad (i = 2, \ldots, n; \quad k = 1, \ldots, m - 1)$$

It may be asserted that vector-function $\{y(x), \xi(x)\}$ belongs to the set $S(D)$, if:

1) Functions $y(x)$ and $\xi_i(x)$ $(i = 2.3, \ldots, n)$ are continuous on the whole x-axis and tend toward a certain limit for $x \to \pm \infty$;

2) Function $y(x)$ has continuous derivatives, including the derivative of the $(m - 1)$ order, all of which tend to zero for $x \to \pm \infty$;

3) for every value $x(-\infty < x < \infty)$ vector $y(x), y'(x), \ldots, y^{(m-1)}(x)$, $\xi_2(x), \ldots, \xi_n(x)$ belongs to the region D;

4) Function $xy(x)$ is not negative.

Let us associate for comparison with every vector function $\{y, \xi\}$ from $S(D)$ the vector function $A\{y, \xi\} = \{y^\times, \xi^\times\}$, which satisfies the following relations:

$$P\left(\frac{d}{dx}\right) y^\times + \varepsilon \beta_0 (y^\times - y^{\times 2}) + \varepsilon \psi(x) = 0$$

$$\xi_i^{\times\prime} - \sum_{j=2}^{m} b_{ij}\,\xi_j^{\times} = \psi_i(x) \qquad (i = 2, 3, \ldots, m)$$

$$\xi_i^{\times} = \psi_i(x) \qquad (i = m+1, \ldots, n)$$

where

$$\psi(x) = -\sum_{k=1}^{m-1} y^{(k)}\varphi_k(y, y', \ldots, y^{(m-1)}, \xi_2, \ldots, \xi_n; \varepsilon) - $$
$$- \varepsilon F_0(y, y', \ldots, \ldots, y^{(m-1)}, \xi_2, \ldots, \xi_n; \varepsilon)$$
$$\psi_i(x) = f_i(y^{\times}) + \varepsilon F_i(y, \xi_2, \ldots, \xi_n; \varepsilon) \qquad (i = 2, 3, \ldots, n)$$

From Theorem 8.1 it follows that these relations, indeed, determine operator A. The Theorem 8.2 and its corollary 8.1 show that for sufficiently small $\varepsilon > 0$ the operator A does not invalidate the vector functions in the region $S(D)$ and that it is compressible relative to the norm

$$\| y, \xi \| = \sup_x \left\{ | y(x) | + \frac{1}{\varepsilon} \sum_{k=1}^{m-1} | y^{(k)}(x) | + \sum_{i=2}^{n} | \xi_i(x) | \right\}$$

with the coefficient of compressibility $\rho < 1$.

Hence it follows that there exists a vector function $\{y(x), \xi(x)\}$ such that $A\{y, \xi\} = \{y, \xi\}$. This function serves as the transitional solution of system (1.1). Theorem (1.1) is proved.

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